Now we are ready to finish the proof of the Serre-Swan theorem.

It is enough to prove, that Γ is essentially surjective on objects, i.e. every finitely generated projective $C(X)$-module $\mathcal{E} \cong e\,C(X)^N$ is isomorphic to $\Gamma(E)$ for some E.

Consider the split short exact sequence

$$0 \longrightarrow \ker e \longrightarrow C(X)^N \longrightarrow \mathcal{E} \longrightarrow 0 .$$

As Γ is full, e is induced by a bundle map

$$\alpha : M \times \mathbb{C}^N \longrightarrow M \times \mathbb{C}^N .$$

Since Γ is faithful such α is unique.

Let E be a subbundle being the image of α. The image of α is indeed a subbundle, since $\alpha^2 = \alpha$, hence $(\mathrm{Id}_{X\times\mathbb{C}^N} - \alpha)^2 = \mathrm{Id}_{X\times\mathbb{C}^N} - \alpha$ and both $\operatorname{rank}\alpha_x$ and $\operatorname{rank}(\mathrm{Id}_{\mathbb{C}^N} - \alpha_x) = N - \operatorname{rank}\alpha_x$ are lower semicontinuous, which means that $x \longmapsto \operatorname{rank}\alpha_x \in \mathbb{N}$ is continuous and therefore constant. Then finally

$$\Gamma(E) = \{\alpha \circ s \mid s \in \Gamma(M \times \mathbb{C}^N)\} = \operatorname{im}(e) = \mathcal{E}. \quad \square$$

The Gelfand-Naimark duality and the Serre-Swan theorem together mean the following.

Corollary. Finitely generated projective modules over commutative C^*-algebras are equivalent to vector bundles over compact Hausdorff spaces.

Moreover, the letter can be described in terms of idempotents in the nonunital algebra

$$\operatorname*{colim}_N M_N(C(X)), \quad \text{where} \quad M_N \xhookrightarrow{\cong} \left\{ \begin{pmatrix} M_N & 0 \\ 0 & 0 \end{pmatrix} \right\} \subset M_{N+1}.$$

Exercise 22. Let G be compact. Let E be a G-equivariant vector bundle on $X = G/H$. Construct a G-equivariant complement E' to E in a trivial vector bundle on X.

Hint: Theorem [Mostow] Every representation $\rho : H \to GL(V)$ of a closed subgroup H of a compact group G is contained in the restriction $\tilde{\rho}|_H$ of some representation $\tilde{\rho} : G \to GL(\tilde{V})$.

Solution. Let $H < G$ be a closed subgroup of G.

$\rho : H \to GL(V)$ representation, $E = G \times^H V$.

By the Mostow theorem

$$\Rightarrow \quad G \times^H V \subseteq G \times^H \tilde{V} \cong G/_H \times \tilde{V}.$$

G compact $\Rightarrow$ it has a Haar measure

$\Rightarrow$ by averaging one can construct an a G-invariant hermitian scalar product on $\tilde{V}$

Then one can take $E' := E^{\perp}$ in $G/_H \times \tilde{V}$

$$\Rightarrow \quad E' \oplus E \cong G/_H \times \tilde{V}. \quad \blacksquare$$

Equivariant Serre-Swan. G compact, X compact G-space.

Theorem. [Equivariant Serre-Swan (after Segal)]
Any G-equivariant vector bundle $E \to X$ is a direct summand of a bundle $V \times X \to X$ for some G-representation $\rho : G \to \mathrm{Aut}(V)$.

Proof. We can use one of the following theorems

Theorem. [Peter-Weyl] The Hilbert space $L^2(G)$ decomposes as an orthogonal sum

$$L^2(G) \cong \bigoplus V_\alpha \otimes V_\alpha^*$$

over all finite dimensional irreducible representations V_α, as a $G \times G$-module.

and

Theorem. [Mostow] If V is a G-Banach space, then elements $v \in V$ such that $\dim(\mathrm{span}(Gv)) < \infty$ are dense.

Remark. For $V = L^2(G)$ the latter is a consequence of the first, but it works also for $C(G)$ with the sup norm.

Proof of Mostow's Theorem. We have a map

$$\begin{aligned} C(G) \times V &\longrightarrow V \\ (a, v) &\longmapsto \int_G a(g) \cdot (gv) \, d\chi(g) =: B(a, v) \end{aligned}$$

If $\int_G a(g)\,d\chi(g) = 1$, then

$$B(a,v) - v = \int_G a(g)(gv - v)\,d\chi(g).$$

Notice that if the support of a in a small neighborhood of identity $e \in G$, then $B(a,v)$ is very close to v.

Also $\quad g'B(a,v) = B(g'a, v)$, if $(g'a)(g) = a(gg')$

Now, fix v and $\varepsilon > 0$ and choose $a \in C(G)$ whose support is very near $e \in G$ and $\int_G a(g)\,d\chi(g) = 1$.

Then $\quad \| B(a,v) - v \| < \varepsilon$.

If $\| \tilde{a} - a \| < \varepsilon$ and G-translates of $\tilde{a}$ span a finite dimensional subspace then

$$\| B(a,v) - B(\tilde{a},v)\| < \sup_{g \in G} \|ga\| \cdot \varepsilon = C \cdot \varepsilon$$

and hence

$$\| B(\tilde{a},v) - v \| \leq \| B(\tilde{a},v) - B(a,v)\| + \| B(a,v) - v\| < (C+1)\varepsilon .$$

Also, $g' B(\tilde{a},v) = B(g'\tilde{a},v) \Rightarrow$ G-translates of $\tilde{a}$ span a finite-dimensional subspace of $C(G)$.

$\Rightarrow B(\tilde{a},v)$ is a desired approximation of v. $\square$

Now we can prove the equivariant Serre-Swan theorem. It suffices to find a finite-dimensional G-invariant subspace $V \subset \Gamma(X, E)$, such that the evaluation map $V \longrightarrow E_x$ is surjective at all $x \in X$.

Choose first $V_0 \subset \Gamma(X, E)$ such that the evaluation map $V_0 \to E_x$ is surjective at all $x \in X$. This is guaranteed by the Serre-Swan theorem.
By the Peter-Weyl theorem we can find a subspace $V_1 \subset \Gamma(X, E)$ which is close to V_0 such that $G V_1$ is contained in a finite-dimensional

G-representation $V \subset \Gamma(X, E)$. Since V_1 is close to V_0 the evaluation maps $V_1 \to E_x$ are surjective as well, since X compact. Therefore all evaluations $V \to E_x$ are surjective. $\square$

Corollary. Elements of $K_G(X)$ can be represented by G-invariant idempotents in $M_\infty(C(X))$.

when two idempotents define the same finitely generated projective right A-module?

Equivalence of idempotents.

$e, f \in M_\infty(A)$ are <u>equivalent</u> if there exist $x, y \in M_\infty(A)$ such that $e = yx$, $f = xy$.

Exercise 23. Show that idempotents in $M_\infty(A)$ together with arrows $e \underset{y}{\overset{x}{\rightleftarrows}} f$ as above form a groupoid.

Exercise 24. Show that

$$e \sim f \iff eA^\infty \cong fA^\infty$$

as right A-modules.

Solution. Let $A^\infty \underset{u}{\overset{u'}{\rightleftarrows}} eA^\infty$, $A^\infty \underset{v}{\overset{v'}{\rightleftarrows}} fA^\infty$

be split epimorphisms s.t. $e = u'u$, $f = v'v$

If $e = yx$, $f = xy$, let $g := vxu'$, $h := uyv'$.

Then $g : eA^\infty \longrightarrow fA^\infty$, $h : fA^\infty \to eA^\infty$ are

mutually inverse iso of right A-modules.

Given mutually inverse isomorphisms

$$g : eA^\infty \longrightarrow fA^\infty \quad , \quad h : fA^\infty \rightarrow eA^\infty$$

one can define $x = u'hv$, $y = v'gu$.

Then $\quad xy = u'hvv'gu = u'hgu = u'u = e$

$\qquad\quad yx = v'guu'hv = v'ghv = v'v = f$. $\square$

Fact. If A is a C^*-algebra then one can assume that idempotents are selfadjoint (then they are called <u>projections</u>) and $v = u^*$, $uu^*u = u$ (then u, v are called <u>partial isometries</u>). The equivalence then is called <u>Murray-von Neumann equivalence</u>.

Exercise 25. Show that a Murray-von Neumann equivalence of projections provides a mutually inverse isomorphisms of right A-modules

$$eA^\infty \underset{y}{\overset{x}{\rightleftarrows}} fA^\infty .$$

Solution. $e = x^*x$, $f = xx^*$

$$\Rightarrow \begin{cases} xe = xx^*x = x , & fx = xx^*x = x \\ x^*f = x^*xx^* = x^* , & ex^* = x^*xx^* = x^* \end{cases}$$

$$\Rightarrow \quad eA^\infty \underset{x^*}{\overset{x}{\rightleftarrows}} fA^\infty \quad \text{mutually inverse}$$

isos of right A-modules. $\square$

Additive structure of the monoid f.g.p-Mod_A.

$$e_1A^\infty \oplus e_2A^\infty \cong \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} A^\infty \quad (\text{note } A^\infty \oplus A^\infty \cong A^\infty).$$

Therefore $e_i \sim f_i$ for $i=1,2$ induces

$$\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \sim \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}.$$

Indeed, if $e_i = y_i x_i$, $f_i = x_i y_i$ then

$$\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}.$$

Therefore it induces an abelian structure on the Grothendieck group. We denote it by $K_0(A)$.

In the G-equivariant case we use the notation $K_0^G(A)$.
Note that $K_G^0(X) = K_0^G(C(X))$ and $K_0^G(A)$ is
a module over $K_0^G(\mathbb{C}) = R(G)$.

Example. $G = \mathbb{T} \implies R(G) = \mathbb{Z}[\hat{\mathbb{T}}] = \mathbb{Z}[t, t^{-1}]$
Therefore for a $\mathbb{T}$-action on X, $K_{\mathbb{T}}^0(X)$ is
a module over $\mathbb{Z}[t, t^{-1}]$.

Noncompact spaces. X locally compact
then $X^+ = X \sqcup \{*\}$ can be given a structure
of a compact space s.t. open neighborhoods
of $*$ are of the form $(X \setminus K) \sqcup \{*\}$, with K compact
in X.

Then $\quad K^0_G(X) := \ker\left(\varepsilon^*: K^0_G(X^+) \to K^0_G(*) = R(G)\right)$

where $\varepsilon : \{*\} \longrightarrow X^+$ a distinguished point of X^+.

Higher K-groups.

$$K^{-n}_G(X) := K^0_G(X \times \mathbb{R}^n).$$

where $n \in \mathbb{N}$ and G acts trivially on $\mathbb{R}^n$.

Theorem. [Bott periodicity]

$$K^i_G(X) \cong K^{i+2}_G(X).$$

Corollary. $K^0_G(\mathbb{R}^n) = \begin{cases} R(G) & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$

Long exact sequence. If $U \subseteq X$ is a closed G-invariant subspace, then there is a long exact sequence

$$\begin{tikzcd}
K^0_G(X,Y) \arrow[r] & K^0_G(X) \arrow[r] & K^0_G(Y) \arrow[d] \\
K^1_G(Y) \arrow[u] & K^1_G(X) \arrow[l] & K^1_G(X,Y) \arrow[l]
\end{tikzcd}$$

Exercise 26. Show that if G acts freely on X, then $K^*(G\backslash X) \cong K^*_G(X)$.

Solution. In fact it is so for $\mathrm{Vect}_G(-)$, hence for $\mathrm{Vect}_G(- \times \mathbb{R}^n)$.

$$\mathrm{Vect}(G\backslash X) \cong \mathrm{Vect}_G(X)$$

$$(F \to G\backslash X) \longmapsto F \times_{G\backslash X} X$$

$$? \longleftarrow (E \to X)$$

$$\Gamma(X,E)^G \longleftarrow \Gamma(X,E)$$

$C(G\backslash X)$-module $\qquad\qquad \shortparallel$

$$e\, C(G\backslash X)^\infty \longleftarrow e\, C(X)^\infty$$

since e is G-invariant.